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Zeros of the Sum of Two Polynomials

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Abstract

In this paper we find bounds for the zeros of the sum of two polynomials whose coefficients are restricted to certain conditions in the framework of Enestrom-Kakeya theorem.

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Introduction

A famous result giving a bound for all the zeros of a polynomial with real positive monotonically decreasing coefficients is the following result known as Enestrom-Kakeya theorem [2]:

Theorem A: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that

$$a_n \geq a_{n-1} \geq ...... \geq a_1 \geq a_0 > 0.$$

(1)

Then all the zeros of P(z) lie in the closed disk $|z| \leq 1$.

If the coefficients are monotonic but not positive, Joyal, Labelle and Rahman [1] gave the following generalization of Theorem A:

Theorem B: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that

$$a_n \geq a_{n-1} \geq ...... \geq a_1 \geq a_0 .$$

Then all the zeros of P(z) lie in the closed disk $|z| \leq \frac{a_n - a_0 + |a_0|}{|a_n|}$.

If P(z) and Q(z) are two polynomials whose coefficients satisfy relations of the type (1), then the question arises whether Theorem A holds good for the sum P(z)+Q(z) of the two polynomials. It is easy to see that if the two polynomials have the same degree, then the conclusion of Theorem A holds good. But if P(z) and Q(z) are not of the same degree, then the result is not true. For example consider the polynomials $P(z)=z^2+z+1$ and $Q(z)=3z+2$. Both satisfy the conditions of Theorem A and both of them have their zeros in $|z| \leq 1$. But $P(z)+Q(z)=z^2+4z+3$, whose zeros are -1, -3 with moduli 1 and 3 so that one zero does not lie in $|z| \leq 1$ and the theorem fails. The same is true of the polynomials $P(z)=5z^2+4z+3$ and $Q(z)=8z+7$. The modulus of each zero of P(z)+Q(z) is 2 >1. In such cases , however, we prove the following result:

Theorem 1: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ and $Q(z)=\sum_{j=0}^{m} b_j z^j$ be polynomials of degrees n and m (n>m) respectively such that

$$a_n \geq a_{n-1} \geq ...... \geq a_1 \geq a_0$$
$$b_m \geq b_{m-1} \geq ...... \geq b_1 \geq b_0 .$$

Then P(z)+Q(z) has all its zeros in the disk

$$|z| \leq \frac{a_n + |b_m| + b_m + |a_0| + |b_0| - a_0 - b_0}{|a_n|}.$$

The bound of Theorem 1for the zeros of P(z)+Q(z) in the first example given above is 7 and $|z| \leq 7$ contains both -1 and -3. Similarly the bound for the second example is 4.2 and $|z| \leq 4.2$ contains both the zeros.

If we take Q(z)=0, Theorem 1 gives Theorem B.

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Proof of Theorem

Proof of Theorem 1: Let n=m+p, p>0 i.e.m=n-p.Consider the polynomial

$$F(z)=(1-z)(P(z)+Q(z))$$
$$=(1-z)[a_n z^n + a_{n-1}z^{n-1}+......+a_{n-p+1}z^{n-p+1}+(a_{n-p}+b_{n-p})z^{n-p}+......$$
$$+(a_{n-p-1}+b_{n-p-1})z^{n-p-1}+......+(a_1+b_1)z+(a_0+b_0)]$$
$$=-a_n z^{n+1}+(a_n-a_{n-1})z^n+(a_{n-1}-a_{n-2})z^{n-1}+......+(a_{n-p+2}-a_{n-p+1})z^{n-p+2}$$
$$+(a_{n-p+1}-a_{n-p}-b_{n-p})z^{n-p+1}+(a_{n-p}+b_{n-p}-a_{n-p-1}-b_{n-p-1})z^{n-p}$$
$$+......+(a_1+b_1-a_0-b_0)z+(a_0+b_0).$$

For $|z|>1$, we have, by using the hypothesis,

$$|F(z)|\geq |a_n||z|^{n+1}-|(a_n-a_{n-1})z^n+(a_{n-1}-a_{n-2})z^{n-1}+......+(a_{n-p+2}-a_{n-p+1})z^{n-p+2}$$
$$+(a_{n-p+1}-a_{n-p}-b_{n-p})z^{n-p+1}+(a_{n-p}+b_{n-p}-a_{n-p-1}-b_{n-p-1})z^{n-p}$$
$$+......+(a_1+b_1-a_0-b_0)z+(a_0+b_0)\ |$$
$$\geq |a_n||z|^{n+1}-|z|^n[|a_n-a_{n-1}|+\frac{|a_{n-1}-a_{n-2}|}{|z|}+\frac{|a_{n-2}-a_{n-3}|}{|z|^2}+......+\frac{|a_{n-p+2}-a_{n-p+1}|}{|z|^{p-2}}$$
$$+\frac{|a_{n-p+1}-a_{n-p}-b_{n-p}|}{|z|^{p-1}}+\frac{|a_{n-p}+b_{n-p}-a_{n-p-1}-b_{n-p-1}|}{|z|^p}+......$$
$$+\frac{|a_1+b_1-a_0-b_0|}{|z|^{n-1}}+\frac{|a_0+b_0|}{|z|^n}]$$
$$>|z|^n[|a_n||z|-\{a_n-a_{n-1}+a_{n-1}-a_{n-2}+......+a_{n-p+2}-a_{n-p+1}+a_{n-p+1}-a_{n-p}$$
$$+|b_{n-p}|+a_{n-p}-a_{n-p-1}+b_{n-p}-b_{n-p-1}+......+a_1-a_0+b_1-b_0+|a_0|+|b_0|]$$
$$=|z|^n[|a_n||z|-\{a_n+|b_{n-p}|+b_{n-p}+|a_0|+|b_0|-a_0-b_0\}]$$

>0 if

$$|a_n||z|-\{a_n+|b_{n-p}|+b_{n-p}+|a_0|+|b_0|-a_0-b_0\}>0$$

i.e.

$$|z|>\frac{1}{|a_n|}(a_n+|b_{n-p}|+b_{n-p}+|a_0|+|b_0|-a_0-b_0).$$

This shows that those zeros of F(z) whose modulus is greater than 1 lie in

$$|z|\leq\frac{1}{|a_n|}(a_n+|b_{n-p}|+b_{n-p}+|a_0|+|b_0|-a_0-b_0).$$

But the zeros of F(z) whose modulus is less than or equal to 1 already satisfy the above inequality. Therefore , it follows that all the zeros of F(z) lie in the disk

$$|z|\leq\frac{1}{|a_n|}(a_n+|b_{n-p}|+b_{n-p}+|a_0|+|b_0|-a_0-b_0).$$

Since the zeros of P(z) are also the zeros of F(z), the result follows.

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